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Scaling Theory of Transient Phenomena Near the Instability Point

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A general scaling theory of transient phenomena is formulated near the instability point for the moments of the relevant intensive macrovariable, for the generating function, and for the probability distribution function. This scaling theory is based on a generalized scale transformation of time. The whole range of time is divided into three regions, namely the initial, scaling, and final regions. The connection procedure between the initial region and the scaling region is studied in detail. This scaling treatment has overcome the difficulty of divergence of the variance for a large time which was encountered in the Ω-expansion, and this scaling theory yields correct values of moments to order unity for an infinite time. Some instructive examples are discussed for the purpose of clarifying the concepts of the scaling theory.

KEY WORDS: Scaling theory; transient phenomena; instability point; macrovariable; nonlinear relaxation and fluctuation; probability distribution function; fluctuation enhancement theorem; system-size expansion, most dominant terms; nonlinear Fokker–Planck equation; Kramers–Moyal equation; laser model.

1. INTRODUCTION

It has been a fascinating but difficult problem to study analytically relaxation from the instability point. In a previous paper[1] (to be referred to as I), a scaling theory for transient phenomena near the instability point has been proposed. The main idea of this scaling theory is to divide the whole range of time into three regions: the *initial region*, in which the linear approximation (or more generally, a perturbational expansion) is valid, the *scaling region*, in which the scaling law holds, and the *final region*, in which the system ap-

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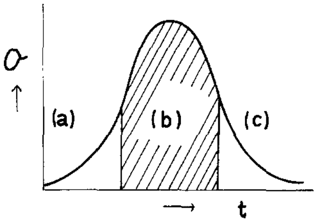

Fig. 1. Fluctuation σ for $\delta \leqslant \epsilon^{\mu}$; (a) initial region, (b) scaling region, (c) final region.

proaches the equilibrium state, as shown in Fig. 1. One of logically simple derivations of the scaling law in the scaling region is to sum up all the most dominant terms in a certain sense. For example, they may take the form $a_n\{\epsilon\omega(t)\}^n$ for small ϵ, where ϵ denotes the inverse system size, i.e., $\epsilon = \Omega^{-1}$, with Ω the system size. In particular, for a typical nonlinear Fokker–Planck equation[1] with the moments $c_1(x) = \gamma x(1 - x^2)$ and $c_2(x) = c$, the most dominant terms of the fluctuation (or moment) $y_2(t, \epsilon)$ of the relevant physical variable x can be calculated as[1]

$$\begin{aligned} y_2(t, \epsilon) \equiv \langle x^2 \rangle &\cong (\epsilon\sigma e^{2\gamma t}) - 3(\epsilon\sigma e^{2\gamma t})^2 + 15(\epsilon\sigma e^{2\gamma t})^3 - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)!!\,\tau^n \\ &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\exp -\frac{x^2}{2}\right) \frac{x^2\tau}{x^2\tau + 1}\,dx \end{aligned} \tag{1}$$

with $\tau = \epsilon\sigma e^{2\gamma t}$, $\sigma = \sigma_0 + \sigma_1$, and $\sigma_1 = c/2\gamma$, for the initial distribution

$$P(x, 0) = \frac{1}{(2\pi\epsilon\sigma_0)^{1/2}} \exp\left(-\frac{x^2}{2\epsilon\sigma_0}\right) \tag{2}$$

The last form of Eq. (1) is nothing but the scaling form $y_{sc}(\tau)$ of $y_2(t, \epsilon)$. The scaling region is specified by the time region in which the scaling time variable τ is of order unity.[1,2] For the above example, the scaling region is given by

$$t \sim (1/2\gamma) \log(1/\epsilon\sigma) \tag{3}$$

It is possible in principle and logically simple to calculate perturbationally the above asymptotic expansion (1), but it is a very much complicated matter to find terms of higher order explicitly. In fact, the expression (1) has been derived in I from the scaling theory. It will be instructive to discuss here how the linear approximation[2] breaks down. It gives an expression

$$y_2^{(lin)}(t, \epsilon) = \epsilon\sigma e^{2\gamma t} - \epsilon\sigma_1 + O(\epsilon^2) \tag{4}$$

This is shown in Fig. 2, together with the scaling solution $y_{sc}(\tau)$. It blows up as t goes to infinity, while the scaling solution approaches the correct equilibrium (or stationary) value $y_2(\infty, \epsilon) = 1 + O(\epsilon)$. This correct approach to

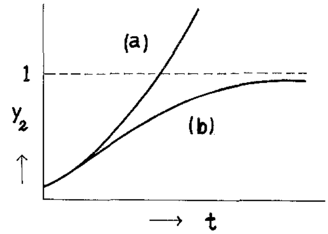

Fig. 2. Schematic time dependence of $y_2(t, \epsilon)$; (a) linear approximation $y_2^{(\mathrm{lin})}(t)$, (b) scaling solution $y_2^{(\mathrm{sc})}(\tau)$.

equilibrium is one of great merits of the scaling theory. These situations will be discussed in general in the present paper more systematically and more explicitly than in I.

One of the main ideas of the scaling theory for transient phenomena is to extract an evolution equation of the scaling function in the scaling region and to connect the solution of it with the dominant part (or scaling part) of the solution in the initial region, at the boundary between the two regions, as shown in Fig. 1. This idea has been performed in I by introducing the following generalized scale transformation of time:

$$\tau = S(t, \epsilon, \delta, \ldots) \tag{5}$$

where δ denotes a deviation of the initial system from the instability point (or asymptotically unstable point for a small ϵ). The evolution equation of the scaling function is evaluated[1,2] asymptotically by *keeping* τ *fixed* in the limit of a small ϵ. This method yields the scaling form[1]

$$f(t, \epsilon, \delta, \ldots) \simeq f_{\mathrm{sc}}(\tau, \delta\epsilon^{-\mu}, \ldots) \tag{6}$$

in the scaling region for physical quantities such as moments, the generating function, and the distribution function, where μ is an appropriate positive exponent. One of the important consequences of the scaling law (6) is that a large enhancement of fluctuation occurs[1] around

$$t_m \sim S^{-1}(1, \epsilon, \delta, \ldots) \tag{7}$$

where S^{-1} denotes the inverse function of $\tau = S(t, \ldots)$. The enhancement factor R for the intrinsic fluctuation $\langle x^2 \rangle_c$ is given by $R \sim \epsilon^{-1}$ in the unstable region, when the initial variance is $\epsilon\sigma_0$ (*fluctuation enhancement theorem*).

The above scaling idea for transient phenomena has also been used to establish generally the anomalous fluctuation theory[2,3] in the extensive region (as shown in Fig. 3), in which the extensive property holds.[3–5]

In Section 2, the physical meaning of the smallness parameter, the existence of a scaling region, and the general scheme of the scaling theory are discussed. The scaling theory (or scaling limit) in the Kramers–Moyal equation is presented generally in Section 3. Furthermore, the scaling solutions of the moments, generating function, and distribution function are given

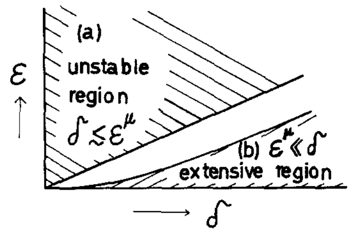

Fig. 3. The ϵ–δ plane; (a) unstable regime $\delta \lesssim \epsilon^{\mu}$, (b) extensive regime $\epsilon^{\mu} \ll \delta$.

explicitly. Some interesting examples are discussed in detail in Section 4. In particular, the relaxation[1] from the unstable state is studied in detail in a typical nonlinear Fokker–Planck equation (i.e., the laser model).

2. PHYSICAL MEANING OF THE SMALLNESS PARAMETER ϵ, THE EXISTENCE OF A SCALING REGION, AND THE GENERAL SCHEME OF THE SCALING THEORY

First we discuss the physical meaning of the smallness parameter ϵ. In the present paper, it denotes the inverse system size $\epsilon = \Omega^{-1}$, where Ω is the volume or the number of particles. As is well known, the fluctuation of a macrovariable X is of order $\Omega^{1/2}$ in a normal situation, while the average value of X is of order Ω. Thus, the relative ratio of the fluctuation to the average motion is $\epsilon^{1/2}$. That is, the smallness parameter ϵ denotes the measure of the effect of fluctuations and consequently it plays the role of the expansion parameter with respect to fluctuation effects.

Next we argue the existence of a scaling region for the case of the relaxation from the unstable point. The average value of the relevant macrovariable does not change in time for a vanishing ϵ if the system is located just at the unstable point at the initial time, because of the lack of fluctuation (or diffusion), as is easily seen from the analogy to classical motion in a potential shown in Fig. 4. The smallness parameter ϵ is also analogous to the Planck constant $\hbar$. The wave function of a quantum mechanical system in a potential shown in Fig. 4 moves toward the stable points even if the initial wave function is of a δ-function type at the unstable point, as far as $\hbar$ is nonvanishing.

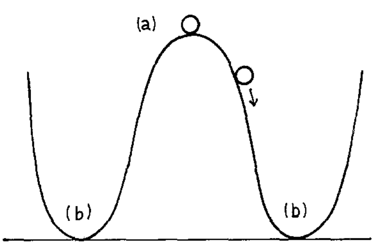

Fig. 4. Classical motion in a potential; (a) unstable point, (b) stable points.

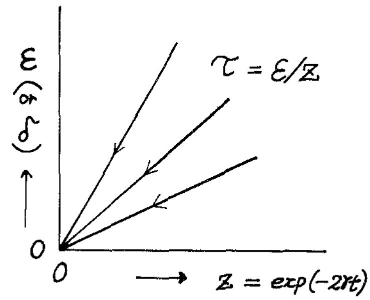

Fig. 5. Physical quantities depend upon the path (or $\tau = \epsilon/z$) near the essential singular point $\epsilon = 0$ (or $\delta = 0$) and $z = 0$.

It is generally concluded from these considerations that physical quantities have an *essential singular point* at $\epsilon = 0$ (or $\delta = 0$) and $z\ (\equiv e^{-2\gamma t}) = 0$ (for a certain constant γ). Therefore, the limiting values of the relevant physical quantities depend upon the path or limiting process, namely upon the ratio $\tau = \epsilon/z$, as shown in Fig. 5. Thus, the physical quantities depend upon the so-called scaling variable τ in the vicinity of the essential singular point $\epsilon = 0$ (or $\delta = 0$) and $z = 0$ ($t \to \infty$). This confirms the existence of a scaling region for a small ϵ (or δ) and a large time t. This is quite analogous to the existence of a scaling region in critical phenomena.

Here a general scaling expansion is formulated using the generalized scale transformation of time (5). We start from the following abstract equation[1]:

$$\frac{\partial}{\partial t} f(t, \epsilon, \delta,\ldots) = \mathscr{L}(t, \epsilon, \delta,\ldots) f(t, \epsilon, \delta,\ldots) \tag{8}$$

where f denotes the distribution function $P(x, t)$, generating function $\Psi(\lambda, t)$, or fluctuations (i.e., moments), and $\mathscr{L}$ is a linear (or nonlinear) operator.[1] As in I, first we apply the scale transformation (5) to (8), and consequently we obtain[1]

$$s(\tau, \epsilon, \delta,\ldots) \frac{\partial}{\partial \tau} f = \mathscr{L}(S^{-1}(\tau, \epsilon, \delta,\ldots), \epsilon, \delta,\ldots) f \tag{9}$$

where

$$s(\tau, \epsilon, \delta,\ldots) = \left[\frac{\partial}{\partial t} S(t, \epsilon, \delta,\ldots)\right]_{t=S^{-1}(\tau,\epsilon,\delta,\ldots)} \tag{10}$$

Keeping τ fixed and $\delta\epsilon^{-\mu}$ fixed, we take the limit $\epsilon \to 0$, and consequently we obtain the evolution equation of the scaling function $f_{sc}(\tau,\ldots)$ of the form

$$\frac{\partial}{\partial \tau} f_{sc} = \mathscr{L}_{sc} f_{sc}; \qquad \mathscr{L}_{sc} = \lim_{\substack{\epsilon\to 0 \\ \tau,\, \delta\epsilon^{-\mu}\text{fixed}}} \{s(\tau, \epsilon, \delta,\ldots)\}^{-1} \mathscr{L}(S^{-1}(\tau, \epsilon, \delta,\ldots), \epsilon, \delta,\ldots) \tag{11}$$

Thus, f takes the scaling form

$$f(t, \epsilon, \delta, \ldots) \simeq f_{\mathrm{sc}}(\tau, \delta\epsilon^{-\mu}, \ldots) \tag{12}$$

in the scaling region. Explicit evaluations of scaling functions together with detailed arguments on the connection procedure of the scaling solution with the solution in the initial region are given in Section 3 for the Kramers–Moyal equation described by

$$\epsilon \frac{\partial}{\partial t} P(x, t) + \mathscr{H}\left(x, \epsilon \frac{\partial}{\partial x}, \epsilon\right) P(x, t) = 0 \tag{13}$$

where

$$\mathscr{H}(x, p, \epsilon) = \int dr\,(1 - e^{-rp}) w(x, r, \epsilon) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} p^n c_n(x, \epsilon) \tag{14a}$$

and

$$c_n(x, \epsilon) = \int r^n w(x, r, \epsilon)\, dr \tag{14b}$$

with the transition probability $w(x, r, \epsilon)$.

3. SCALING THEORY IN THE KRAMERS–MOYAL EQUATION

In this section we discuss explicitly and systematically how to connect scaling solutions in the scaling region with those in the initial region for the Kramers–Moyal equation. In order to find heuristically a correct connection procedure between the initial and scaling regions, we start from the equations of motion for moments $\{\langle x^n \rangle\}$ instead of the distribution function $P(x, t)$, because it is easy to evaluate asymptotically the order of magnitude of moments for a small ϵ. After a correct connection procedure has been found for moments, it is easily transferred to the connection procedures of the generating function and distribution function, so that the scaling functions thus connected may give the same expressions of moments as those obtained from the equations of motion for moments. The logical steps of these connection procedures for moments, the generating function, and the distribution function are illustrated in Fig. 6. Once these connection procedures have been established, the distribution function is the most convenient among them for practical applications of the scaling theory, because the evolution equation of the scaling distribution function for the Kramers–Moyal equation is expressed by the drift equation (i.e., a linear partial differential equation of first order), and consequently it can be solved generally, as will be discussed later.

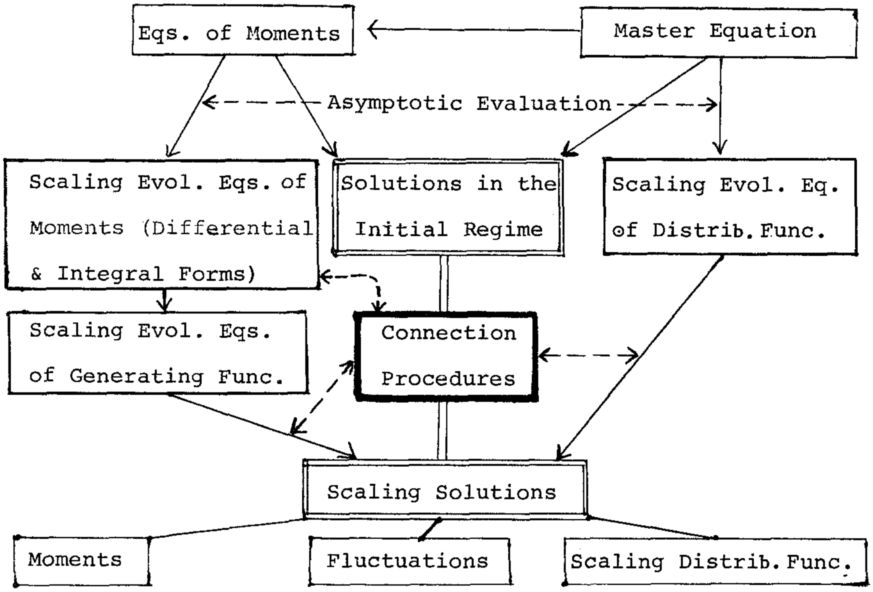

Fig. 6. Diagram illustrating the scheme of the scaling theory.

3.1. Scaling Theory Based on Moments

As was proven in the previous paper,[1] the average value $\langle Q(x)\rangle$ of an arbitrary function $Q(x)$ defined by

$$\langle Q(x)\rangle = \int Q(x)P(x, t)\, dx \tag{15}$$

satisfies the equation

$$\epsilon \frac{d}{dt}\langle Q(x)\rangle + \left\langle \mathscr{H}^*\left(x, \epsilon\frac{\partial}{\partial x}, \epsilon\right) Q(x)\right\rangle = 0 \tag{16}$$

for the Kramers–Moyal equation (13), where $\mathscr{H}^*$ is the adjoint operator of $\mathscr{H}$ defined by

$$\mathscr{H}^*(x, p, \epsilon) \equiv -\sum_{n=1}^{\infty} \frac{1}{n!} c_n(x, \epsilon)p^n \tag{17}$$

and we have performed partial integrations iteratively, assuming that $x^nP(x, t)$ vanishes at the boundaries of x for any positive integer n. For simplicity, we discuss here a symmetric situation in which $c_{2n}(-x, \epsilon) = c_{2n}(x, \epsilon)$ and $c_{2n-1}(-x, \epsilon) = -c_{2n-1}(x, \epsilon)$. Our main results on connection procedures hold irrespective of this restriction. From (16), it is shown that the moment $y_{2n}(t)$ defined by

$$y_{2n}(t) = \langle x^{2n} \rangle \tag{18}$$

satisfies the following infinitely coupled equations:

$$\frac{d}{dt} y_{2n}(t) = 2n\gamma y_{2n} + f_{2n}(\{y_{2j}\}) + \epsilon g_{2n}(\{y_{2j}\}, \epsilon) \tag{19}$$

where

$$f_{2n}(\{y_{2j}\}) \equiv 2n\langle c(x)x^{2n-1} \rangle \tag{20a}$$

$$g_{2n}(\{y_{2j}\}, \epsilon) \equiv 2n\langle c(x, \epsilon)x^{2n-1} \rangle + \sum_{k=2}^{\infty} \frac{\epsilon^{k-2}}{k!} \left\langle c_k(x, \epsilon) \frac{d^k}{dx^k} x^{2n} \right\rangle \tag{20b}$$

with

$$c(x) \equiv c_1(x) - \gamma x, \qquad c_1(x) = c_1(x, 0), \qquad \gamma = c_1'(0) > 0 \tag{21}$$

$$\epsilon c(x, \epsilon) \equiv c_1(x, \epsilon) - c_1(x)$$

Here, without loss of generality, we have assumed that $x = 0$ is an asymptotically unstable point for a small ϵ, i.e., $c_1(0) = 0$. In the following we assume for simplicity that all $\{c_k(x)\}$ are analytic at $x = 0$, i.e., they are expandable in Taylor series of x. The initial distribution function is also assumed to be given by (2) for brevity. (It is easy to extend our arguments to a more general initial condition.) Consequently, the initial values of y_{2n} are given by

$$y_{2n}(0) = b_n(\epsilon\sigma_0)^n; \qquad b_n = (2n - 1)!! \tag{22}$$

It is easily shown that $y_{2n}(t)$ takes the asymptotic form

$$y_{2n}(t) = b_n\{\epsilon\sigma(t)\}^n + O(\epsilon^{n+1}); \qquad \sigma(t) = \sigma e^{2\gamma t} - \sigma_1 \tag{23}$$

in the initial region, where

$$\sigma = \sigma_0 + \sigma_1 \qquad \text{and} \qquad \sigma_1 = c_2(0, 0)(2\gamma)^{-1} \tag{24}$$

This is derived from the following integral equation, which is equivalent to (19):

$$y_{2n}(t) = e^{2n\gamma t}\left[\int_0^t e^{-2n\gamma s} f_{2n}(\{y_{2j}(s)\})\, ds + \epsilon \int_0^t e^{-2n\gamma s} g_{2n}(\{y_{2j}(s)\}, \epsilon)\, ds + y_{2n}(0)\right] \tag{25}$$

or more directly from the following distribution function in the initial region:

$$P_{\text{ini}}(x, t) = \frac{1}{[2\pi\epsilon\sigma(t)]^{1/2}} \exp\left(-\frac{x^2}{2\epsilon\sigma(t)}\right) \tag{26}$$

which is the solution of the linearized Fokker–Planck equation

$$\frac{\partial P_{\text{ini}}}{\partial t} = \left[-\frac{\partial}{\partial x}(\gamma x) + \frac{1}{2}\epsilon c_2(0, 0)\frac{\partial^2}{\partial x^2}\right]P_{\text{ini}} \tag{27}$$

In Appendix A, we give the derivation of (23) from (25). *This derivation confirms the validity of the simplification of the Kramers–Moyal equation to the linearized Fokker–Planck equation (27) in the initial region.* It should be noted that the dominant part of $y_{2n}(t)$ in (23) for a large t near the boundary between the initial and scaling regions takes the following scaling form:

$$y_{2n}(t) = b_n(\epsilon\sigma e^{2\gamma t})^n + \cdots = b_n\tau^n + \cdots \tag{28}$$

where

$$\tau = \epsilon\sigma e^{2\gamma t} = \epsilon(\sigma_0 + \sigma_1)e^{2\gamma t} \tag{29}$$

This expression for τ gives a typical example of the generalized scale transformation of time (5). As is seen from the above argument, an appropriate choice of the scale transformation S in (5) can be made mostly by studying the asymptotic behavior of the dominant part of the solution in the initial region.

In the scaling region, the $\{y_{2n}\}$ are shown from (11) to satisfy the following coupled evolution equations:

$$\tau\frac{d}{d\tau}y_{2n} = ny_{2n} + (2\gamma)^{-1}f_{2n}(\{y_{2j}\}) \tag{30}$$

Since $f_{2n}(\{y_{2j}\})$ does not contain y_{2n}, the differential equation (30) has a *Poincaré nodal point* at $y_{2n} = 0$ and $\tau = 0$. The solution of (30) has an indefinite term of the form $c_n\tau^n$ with an arbitrary constant c_n, for the initial condition that $y_{2n} = 0$ for all n at $\tau = 0$ [which is automatically contained in (30) because $f_{2n}(\{0\}) = 0$]. Therefore, it is fortunately possible to connect smoothly the solution of (30) with the *scaling* dominant part (26) in the initial region. Thus, we can determine uniquely the scaling functions of the moments $\{y_{2n}^{(\text{sc})}(\tau)\}$. In fact, Eq. (30) is transformed into the following integral equation:

$$y_{2n}(\tau) = \tau^n\int_0^\tau (2\gamma\xi^{n+1})^{-1}f_{2n}(\{y_{2j}\})\,d\xi + c_n\tau^n \tag{31}$$

From the condition that $y_{2n}(\tau) = b_n\tau^n + \cdots$ for a small τ, we obtain that $c_n = b_n$. Therefore, the $\{y_{2n}\}$ satisfy

$$y_{2n} = \tau^n\int_0^\tau (2\gamma\xi^{n+1})^{-1}f_{2n}(\{y_{2j}\})\,d\xi + b_n\tau^n \tag{32}$$

with (22).

In order to justify more rigorously the above connection procedure, we make use of the formulation in an *integral form*[1] instead of the above *differential form*. That is, we start from the integral equations (25), and extract asymptotically dominant scaling evolution equations for $\{y_{2n}(\tau)\}$ in integral forms. Then, the matching between the initial and scaling regions is implemented automatically in the course of the asymptotic evaluation. Now, we make the scale transformation (29) in (25), and consequently we obtain

$$y_{2n}(\tau) = \tau^n\left[\int_{\tilde{\epsilon}}^{\tau} f_{2n}(\{y_{2j}(\xi)\})(2\gamma\xi^{n+1})^{-1}\,d\xi + \epsilon\int_{\tilde{\epsilon}}^{\tau} g_{2n}(\{y_{2j}(\xi)\},\epsilon)(2\gamma\xi^{n+1})^{-1}\,d\xi + \tilde{\epsilon}^{-n}y_{2n}(0)\right] \tag{33}$$

with $\tilde{\epsilon} = \epsilon\sigma$ and (22). Here, *keeping* τ *fixed*, we take the limit $\epsilon \to 0$. Thus, we find that

$$\lim_{\substack{\epsilon\to 0\\ \tau\ \text{fixed}}}\left[\epsilon\int_{\tilde{\epsilon}}^{\tau} g_{2n}(\{y_{2j}(\xi)\},\epsilon)(2\gamma\xi^{n+1})^{-1}\,d\xi + \tilde{\epsilon}^{-n}y_{2n}(0)\right] = b_n \tag{34}$$

Note that the integral on the left-hand side of (34) diverges proportionally to ϵ^{-1} for a small ϵ and consequently that the first term in (34) makes a finite contribution. In this sense, the diffusion effect coming from the c_2 term of (14a) is partially included in our scaling theory. This is reflected in the definition of the scaling variable τ as is seen in (29). Equivalently, this effect is produced through the connection procedure. For the derivation of (34), see Appendix B. Thus, we arrive at the integral equation (32), taking the limit $\epsilon \to 0$ in (33) *for* τ *fixed*. This gives a perfect justification of the connection procedure adopted above for the formulation in a differential form.

It is possible[1] to solve the coupled integral equations (32) in asymptotic series in τ. It is, however, more convenient to find the solutions in closed form by the help of the generating function, as will be discussed in the following subsection.

3.2. Generating Function Formalism

Since we are now discussing a symmetric situation, it is convenient to introduce here the following generating function:

$$\hat{\Psi}(\lambda, t) = \langle\exp(\lambda x^2)\rangle \tag{35}$$

instead of the ordinary generating function[1,4] defined by $\Psi(\lambda, t) = \langle\exp(\lambda x)\rangle$. It is easily shown in two ways that the scaling generating function $\hat{\Psi}_{\mathrm{sc}}(\lambda, \tau)$ satisfies an evolution equation of the form

$$2\gamma\tau\frac{\partial}{\partial\tau}\hat{\Psi}_{\mathrm{sc}}(\lambda, \tau) = \lambda\hat{c}_1\left(\frac{\partial}{\partial\lambda}\right)\hat{\Psi}_{\mathrm{sc}}(\lambda, \tau) \tag{36}$$

where

$$\hat{c}_1(x) = 2\sqrt{x}\, c_1(\sqrt{x}) \tag{37}$$

A simple way of deriving (36) is to rewrite the coupled evolution equations (30) in the generating function formalism to get the result (36) with the use of (35). Another means of derivation is to apply the general scaling procedure (11) to the evolution equation of $\hat{\Psi}(\lambda, t)$, which is shown in Appendix C to take the form

$$\epsilon \frac{\partial}{\partial t} \hat{\Psi}(\lambda, t) + \hat{\mathscr{H}}^*\left(\frac{\partial}{\partial \lambda}, \epsilon\lambda, \epsilon\right)\hat{\Psi}(\lambda, t) = 0 \tag{38}$$

where $\hat{\mathscr{H}}^*$ is the adjoint operator of $\hat{\mathscr{H}}$ defined by (C.3) and is given by

$$\hat{\mathscr{H}}^*(\xi, p, \epsilon) = -\sum_{n=1}^{\infty} \frac{1}{n!} p^n \hat{c}_n(\xi, \epsilon) \tag{39}$$

and

$$\hat{c}_n(\xi, \epsilon) = 2^n \xi^{n/2} c_n(\sqrt{\xi}) \tag{40}$$

The general formula (11) yields (36) for the present case (38).

Now, the scaling part of the generating function in the initial region is found, from (26) or (28), to be

$$\hat{\Psi}_{\text{ini}} \cong \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} b_n \tau^n = \frac{1}{(2\pi\tau)^{1/2}} \int_{-\infty}^{\infty} \left[\exp\left(-\frac{x^2}{2\tau} + \lambda x^2\right)\right] dx = (1 - 2\lambda\tau)^{-1/2} \tag{41}$$

We solve the scaling evolution equation (36) so that its solution may be connected smoothly with (41). The result thus obtained is given by

$$\hat{\Psi}_{\text{sc}}(\lambda, \tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\exp -\frac{x^2}{2}\right) \exp\{\lambda[f^{-1}(\sqrt{\tau}x)]^2\}\, dx \tag{42}$$

where $f^{-1}(y)$ is the inverse function of $f(x)$ defined by

$$f(x) = \exp \int_{a_0}^{x} \frac{\gamma}{c_1(\xi)}\, d\xi = x + \cdots \tag{43}$$

For details, see Appendix D.

Thus, the moments $\{y_{2n}(\tau)\}$ are expressed by the integrals

$$\begin{aligned} y_{2n}(\tau) = \langle x^{2n} \rangle &= \left[\frac{\partial^n}{\partial \lambda^n} \hat{\Psi}_{\text{sc}}(\lambda, \tau)\right]_{\lambda=0} \\ &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\exp -\frac{x^2}{2}\right) \{f^{-1}(\sqrt{\tau}x)\}^{2n}\, dx \end{aligned} \tag{44}$$

Note that

$$y_{2n}(\tau) \to x_e^{2n} \qquad \text{for} \quad \tau \to \infty \tag{45}$$

where x_e is the stationary point of x defined by

$$c_1(x_e) = 0 \qquad \text{or} \qquad x_e = f^{-1}(\infty) \tag{46}$$

and that

$$y_{2n} = b_n \tau^n + \cdots \qquad \text{for small } \tau \tag{47}$$

3.3. Scaling Theory Based on the Distribution Function

In this subsection we find the scaling distribution function, so that it may be connected with the following scaling part of the distribution function in the initial region:

$$P_{\text{ini}}^{(\text{sc})}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} \exp\left(-\frac{x^2}{2\tau}\right) \tag{48}$$

which is obtained from (26) with the use of $\sigma(t)$ defined by (23). This gives the asymptotic expression (28) for the moment y_{2n}, and also satisfies the relation

$$\Psi_{\text{ini}}^{(\text{sc})}(\lambda, \tau) = \int_{-\infty}^{\infty} [\exp(\lambda x^2)] P_{\text{ini}}^{(\text{sc})}(x, \tau)\, dx = (1 - 2\lambda\tau)^{-1/2} \tag{49}$$

as it should.

Following the general theory presented in Section 2, the scaling distribution function is governed by the following *drift equation*:

$$\frac{\partial}{\partial t} P_{\text{sc}} + \frac{\partial}{\partial x} c_1(x) P_{\text{sc}} = 0 \tag{50}$$

or equivalently

$$2\gamma\tau \frac{\partial}{\partial \tau} P_{\text{sc}} + \frac{\partial}{\partial x} c_1(x) P_{\text{sc}} = 0 \tag{51}$$

The general solution of (50) or (51) is given by

$$P_{\text{sc}} = \frac{1}{c_1(x)} \phi\left(\frac{1}{\gamma} \log f(x) - t\right) = \frac{1}{(2\pi\tau)^{1/2}} f'(x) \psi(f^2(x)/\tau) \tag{52}$$

where $\phi(y)$ [or $\psi(y)$] is an arbitrary function of y, and $f(x)$ is defined by (43).

There are two equivalent methods of determining the arbitrary function ϕ or ψ so that the solution (52) may be connected smoothly with the dominant scaling solution in the initial region:

(a) One of the simplest methods is to connect (52) with (48) for small τ near $x = 0$. Noting that $f(x) = x + \cdots$ for small x, we obtain

$$P_{sc}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} f'(x) \exp\left(-\frac{f^2(x)}{2\tau}\right) \tag{53}$$

This confirms the expression for Ψ_{sc} in (42) as follows:

$$\begin{aligned}\Psi_{sc}(\lambda, \tau) &= \int_{-\infty}^{\infty} [\exp(\lambda x^2)] P_{sc}(x, \tau)\, dx \\ &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left[\exp\left\{-\frac{1}{2}\xi^2\right\}\right] \exp\{\lambda[f^{-1}(\sqrt{\tau}\xi)]^2\}\, d\xi \end{aligned} \tag{54}$$

where we have made a transformation of variables $\xi = \tau^{-1/2} f(x)$. This yields the justification of the above connection procedure.

(b) An equivalent connection procedure is to solve the "drift" equation (51) with the *modified initial condition*

$$P_i(x) = P_{\text{ini}}^{(\text{sc})}(x, \tau_i) = \frac{1}{(2\pi\tau_i)^{1/2}} \exp\left(-\frac{x^2}{2\tau_i}\right) \tag{55}$$

at $\tau = \tau_i$. More explicitly, we may put $\tau_i = \epsilon\sigma = \epsilon(\sigma_0 + \sigma_1)$, which corresponds to $t_i = 0$, for $\tau_i = \epsilon\sigma \exp(2\gamma t_i)$. As will be seen later, one of the essential points of our scaling theory is that the scaling solution can be determined *uniquely*, irrespective of how we choose τ_i as long as $\tau_i \ll \tau$. With the use of the general solution (52), the solution of (50) or (51) with the initial condition (55) is easily given in the form

$$\begin{aligned} P(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} f'(x) &\left\{ f'\left(f^{-1}\left(\left[\frac{\tau_i}{\tau}\right]^{1/2} f(x)\right)\right)\right\}^{-1} \\ &\times \exp\left[-(2\tau_i)^{-1}\left\{f^{-1}\left(\left[\frac{\tau_i}{\tau}\right]^{1/2} f(x)\right)\right\}^2\right] \end{aligned} \tag{56}$$

Insofar as we are concerned with the scaling form, expression (56) is simplified to

$$P_{sc}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} f'(x) \exp\left(-\frac{f^2(x)}{2\tau}\right) \tag{57}$$

by taking the limit $\tau_i\tau^{-1} \to 0$ for fixed τ with the use of the property (43). Thus, we obtain the same result as (53). Note that the variance of $P_i(x)$ is $\sigma_0 + \sigma_1$ instead of σ_0. This replacement is essential for the case of $\sigma_0 = 0$ (i.e., for the case that the initial distribution function is a δ function). Otherwise, the solution of the drift equation is completely classical or deterministic, as was discussed in I.

For an arbitrary initial distribution function

$$P_i(x) = g(x, \tau_i) \tag{58}$$

the solution of (51) is given by

$$P(x, \tau) = \left(\frac{\tau_i}{\tau}\right)^{1/2} f'(x)\left\{f'\left(f^{-1}\left(\left[\frac{\tau_i}{\tau}\right]^{1/2} f(x)\right)\right)\right\}^{-1} g\left(f^{-1}\left(\left[\frac{\tau_i}{\tau}\right]^{1/2} f(x), \tau_i\right)\right) \tag{59}$$

This gives the explicit result (56) for the initial distribution (56), as it should.

The scaling distribution function can be rewritten as

$$P_{sc}(x, \tau) = [1/(2\pi\tau)^{1/2}] \exp \phi(x, \tau) \tag{60}$$

where

$$\phi(x, \tau) = -(1/2\tau) f^2(x) + \log f'(x) \tag{61}$$

Therefore, the most probable path $y(t)$ is given by the solution of the equation

$$[f(y)]^2 = \tau[1 - \gamma^{-1} c_1'(y)] \tag{62}$$

However, it should be noted that this most probable path $y(t)$ is not so useful in evaluating physical quantities asymptotically, because the variance in this scaling region is very large (i.e., it is of order unity) compared to that in the extensive regime (i.e., initial or final region). It is useful only for discussing the overall features of the relaxation of the distribution function. In fact, with the use of the most probable path $y(t)$, we can define a characteristic time t_0 [or $\tau_0 = \epsilon\sigma \exp(2\gamma t_0)$], which is called the *transition time* from a single peak [namely $y(t) \equiv 0$] to double peaks [namely $y(t) > 0$]. It is determined from

$$\tau_0 = \lim_{y \to 0} f^2(y)[1 - \gamma^{-1} c_1'(y)]^{-1} \tag{63}$$

Therefore, we have

$$t_0 = \frac{1}{2\gamma} \log\left(\frac{\tau_0}{\epsilon\sigma}\right) \sim -\frac{1}{2\gamma} \log \epsilon \tag{64}$$

That is, the transition time is of order $\log(1/\epsilon)$. The most probable path shows the singularity:

$$y(\tau) \cong \mp a(\tau - \tau_0)^{1/2}; \qquad a > 0 \tag{65}$$

near (and after) the transition time, as shown in Fig. 7. Here, a is given by

$$a^2 = -12 c_1^{(3)}(0)[\tau_0 c_1^{(5)}(0) + 8\gamma f^{(3)}(0)]^{-1}. \tag{66}$$

Furthermore, if we define "Gaussian variance" σ_G by

$$P_{sc}(x, \tau) \sim \exp\{-[x - y(t)]^2/2\sigma_G\} \tag{67}$$

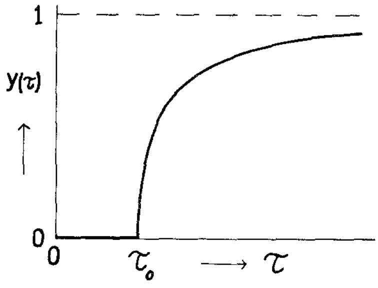

Fig. 7. Time dependence of the most probable path $y(\tau)$.

near the most probable path $y(t)$, then it shows the singularity

$$\sigma_G = \frac{\tau}{1 - \tau f^{(3)}(0)} \cong \frac{{\tau_0}^2}{\tau_0 - \tau} \quad \text{for} \quad \tau < \tau_0 \tag{68}$$

and also

$$\sigma_G = -\left(\frac{\partial^2 \phi}{\partial x^2}\right)^{-1}_{x=y(t)} \sim \frac{1}{\tau - \tau_0} \quad \text{for} \quad \tau > \tau_0 \tag{69}$$

near the transition time, as shown in Fig. 8. Although this singularity may not be observable, it will be of great interest in that it shows an *instability with respect to time* corresponding to the large enhancement of fluctuation and that these singularities are completely analogous to those of the Landau theory on phase transitions in equilibrium.

These situations were already partly discussed in I for a simple model and will be discussed more explicitly in Section 4.

To summarize the main results of this section, as is illustrated in Fig. 6, we have started from the equations of motion for moments to find the correct connection procedure between the initial and scaling regions, and have transformed it into the formulation of the generating function and the distribution function. The connection procedure for the distribution function may be the most convenient for practical purposes.

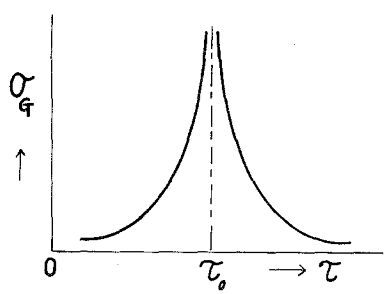

Fig. 8. Gaussian variance σ_G.

4. EXAMPLES

It will be instructive to discuss here several examples of the general arguments given in Sections 2 and 3. The first two are examples of the scaling theory, and the third is concerned with the systematic scaling expansion, which yields an alternative explanation of the scaling theory.

4.1. Unstable Gaussian Distribution—Linear Fokker–Planck Equation

The simplest example of the relaxation near the *unstable* point is the linear Fokker–Planck equation of the form

$$\frac{\partial}{\partial t}P(x,t) = \left[-\frac{\partial}{\partial x}(\gamma x) + \frac{1}{2}\epsilon c\frac{\partial^2}{\partial x^2}\right]P(x,t) \tag{70}$$

As is well known, the solution of this equation with an initial distribution function of the form

$$P(x,0) = \frac{1}{(2\pi\epsilon\sigma_0)^{1/2}}\exp\left[-\frac{(x-\delta)^2}{2\epsilon\sigma_0}\right] \tag{71}$$

is given by

$$P(x,t) = \frac{1}{[2\pi\epsilon\sigma(t)]^{1/2}}\exp\left[-\frac{(x-\delta e^{\gamma t})^2}{2\epsilon\sigma(t)}\right] \tag{72}$$

where

$$\sigma(t) = \sigma e^{2\gamma t} - \sigma_1, \qquad \sigma = \sigma_0 + \sigma_1, \qquad \sigma_1 = (2\gamma)^{-1}c \tag{73}$$

It is easily seen for the time region $\sigma e^{2\gamma t} \gg \sigma_1$ that $P(x,t)$ has the following scaling property:

$$P(x,t) \simeq P_{sc}(x,\tau,\delta\epsilon^{-\mu}) = \frac{1}{(2\pi\tau)^{1/2}}\exp\left(-\frac{[x-\delta\epsilon^{-\mu}(\tau/\sigma)^{1/2}]^2}{2\tau}\right) \tag{74}$$

where $\tau = \epsilon\sigma e^{2\gamma t}$ and $\mu = 1/2$. It should be remarked that the whole time region except the initial region happens to be the scaling region for this special model. Thus, this gives an exactly soluble, simple example of the scaling property.

4.2. Laser Model

This model is described by the following typical nonlinear Fokker–Planck equation[1–4,6]:

$$\frac{\partial}{\partial t}P(x,t) = \left[-\frac{\partial}{\partial x}c_1(x) + \epsilon\frac{\partial^2}{\partial x^2}\right]P(x,t) \tag{75}$$

with

$$c_1(x) = \gamma x(1 - x^2); \qquad \gamma > 0 \tag{76}$$

Here, without loss of generality we have put $x_e = \pm 1$. The function $f(x)$ defined by (43) takes the form[2]

$$f(x) = \frac{x}{(1 - x^2)^{1/2}} \quad \text{and} \quad f^{-1}(y) = \frac{y}{(y^2 + 1)^{1/2}} \tag{77}$$

for the present model. Thus, the general formula (56) yields

$$P(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} \left(1 - x^2 + \frac{\tau_i}{\tau} x^2\right)^{-3/2} \exp\left(-\frac{x^2}{2[\tau(1 - x^2) + \tau_i x^2]}\right) \tag{78a}$$

which is reduced to the scaling solution

$$P_{\mathrm{sc}}(x, \tau) = \frac{1}{(2\pi\tau)^{1/2}} \exp\left[-\frac{x^2}{2\tau(1 - x^2)} - \frac{3}{2}\log(1 - x^2)\right] \tag{78b}$$

for $\tau_i \ll \tau$, as was already reported.[1] The transition time τ_0 from a single peak to double peaks is found from (78b) to be $\tau_0 = 1/3$, which yields

$$t_0 = -(2\gamma)^{-1}\log(3\sigma\epsilon) \tag{79}$$

The most probable path $y(t)$ has the following singularity[1]:

$$y(\tau) = \left(1 - \frac{1}{3\tau}\right)^{1/2} = [1 - (3\sigma\epsilon)^{-1}e^{-2\gamma t}]^{1/2}$$

$$\simeq \sqrt{3}(\tau - \tau_0)^{1/2} \propto (t - t_0)^{1/2} \tag{80}$$

near and after the transition time, as has been discussed generally in Section 3 (see Fig. 7).

The moment y_{2n} is given by[1]

$$y_{2n} \equiv \langle x^{2n} \rangle = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\exp -\frac{x^2}{2}\right)\left(\frac{x^2\tau}{x^2\tau + 1}\right)^n dx \tag{81}$$

from the general formula (44) with the use of (77). In particular, the second moment $y_2(t, \epsilon)$ is expanded in an asymptotic series of τ, as has been demonstrated in (1).

Physically, the temporal evolution of the fluctuation is determined in the initial region by the cooperative effect between the linear drift term and the diffusion term. For the scaling region, it is governed mainly by the drift term, as shown in Fig. 9, in which arrows denote the change of velocity of the distribution function due to the drift term. In the final region, the drift and diffusion terms become equally important, and consequently the equilibrium distribution function is determined from their balance, as shown in Fig. 10.

The present scaling result is consistent qualitatively with Saito's numerical results[7] based on the double-Gaussian approximation.[8] Recently, Tomita

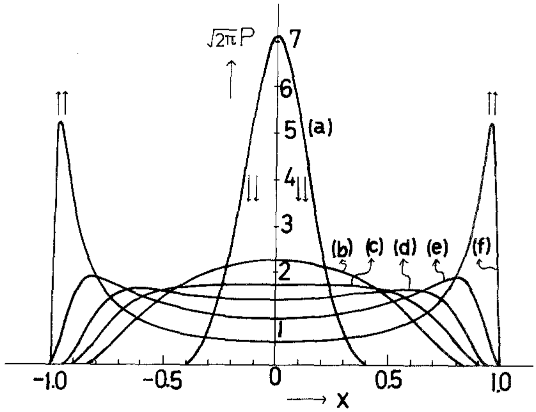

Fig. 9. Change of the distribution function; (a) $\tau = 0.02$, (b) $\tau = 0.2$, (c) $\tau = \tau_0 = 1/3$, (d) $\tau = 0.5$, (e) $\tau = 1$, and (f) $\tau = 4$, where $\tau = \sigma\epsilon e^{2\gamma t}$; the arrows show the direction of the change of velocity of $P(x, t)$.

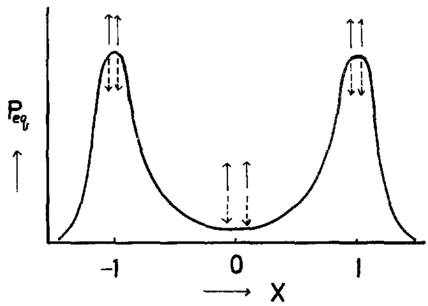

Fig. 10. Equilibrium distribution function due to the balance between the drift force (upward arrows) and diffusion force (downward arrows).

et al.[9] have applied the present asymptotic evaluation method and connection procedure of our scaling theory[1] to the same model (75), on the basis of a "quantum mechanical" formulation, namely using a Schrödinger equation equivalent to (75). They have obtained the decay process of the wave function, which corresponds essentially to ours,[1] as it should.

4.3. Fokker–Planck Equation with a Linear Drift Term and a Nonlinear Diffusion Term

In contrast with the above laser model, we consider here the Fokker–Planck equation with a *linear drift term* and a *nonlinear diffusion term*:

$$\frac{\partial}{\partial t} P(x, t) = \left[-\frac{\partial}{\partial x}(\gamma x) + \frac{\epsilon}{2}\frac{\partial^2}{\partial x^2} c_2(x) \right] P(x, t) \tag{82}$$

Here, we assume for simplicity that $c_2(0) > 0$. In order to apply our scaling theory to this specific model, we calculate first the auxiliary function $f(x)$ defined by (43) to get the result $f(x) = x$. Consequently, from (57) the scaling function takes the Gaussian form

$$P_{sc}(x, \tau) = [1/(2\pi\tau)^{1/2}] \exp(-x^2/2\tau); \qquad \tau = \epsilon\sigma e^{2\gamma t} \tag{83}$$

for the initial distribution function (2), where $\sigma = \sigma_0 + \sigma_1$ and $\sigma_1 = c_2(0)(2\gamma)^{-1}$. That is, our scaling theory becomes equivalent to the linearization of the FP equation for this specific example. By the linearization, we mean the replacement of the second moment $c_2(x)$ by the constant part $c_2(0)$. It is clear that the solution of the FP equation thus linearized has the scaling property, as in Section 4.1. Thus, we do not need explicitly the connection procedure for this particular case.

It should be noted that the approximation or linearization implemented by Glauber and Haake[(10)] in discussing fluctuations of superradiance corresponds to a special case of our scaling theory for the specific model (82) in the above sense, *although the important concept of the connection between two regions was not introduced in their argument.* Thus, our scaling theory gives a justification and limitation of the treatment by Glauber and Haake[(10)] on superradiance.

A systematic scaling expansion for this model (82) will be obtained by treating the "nonlinear" term

$$\frac{1}{2}\epsilon \frac{\partial^2}{\partial x^2} [c_2(x) - c_2(0)]P(x, t) \tag{84}$$

as a perturbation. An example of this perturbational expansion has been given by Narducci and Bluemel[11] for the case of superradiance.

5. CONCLUDING REMARKS

The scaling theory of transient phenomena near the instability point has been presented. The systematic scaling expansion from the scaling limit will be reported in the near future. The second term of the scaling expansion (which is of order ϵ) has a particular importance as the time goes to infinity, because the variance or fluctuation in the final region (or near equilibrium) is of order ϵ. That is, such a fluctuation may be calculated from the first correction to the scaling limit.

There may be another method to study the fluctuation and relaxation near the equilibrium state, namely one may find a connection procedure between the scaling region and the final region, just as for the initial and scaling regions. One of the simplest connection procedures is to connect, at the boundary of the two regions, the most probable path $y(t)$ and Gaussian variance $\sigma_G(t)$ around it obtained in the scaling region with those of the final region, which are the solutions of the following evolution equation[3,4,12]:

$$\frac{d}{dt}y(t) = c_1(y(t)), \qquad \frac{d}{dt}\sigma(t) = 2c_1'(y(t))\sigma(t) + c_2(y(t)) \tag{85}$$

This connection procedure will be useful in analyzing experimental data near the instability point.

As illustrated in Fig. 6, the present derivation of the scaling theory, in particular the connection procedure, has been implemented on the basis of the equations of motion for moments, because it is much easier to evaluate their order asymptotically for small ϵ, compared to the distribution function. However, once the connection procedure has been found explicitly, it is more convenient to make use of the distribution function in studying the scaling property.

The scaling solutions (57) and (78b) can be used even in the initial region if we replace τ by $\epsilon\sigma(t)$ defined by (23).

Some applications of the scaling expansion, for example, to superradiance,[13] will be reported elsewhere.

The present formulation of scaling theory for a single intensive variable x will be extended in the future to multicomponent systems, and more generally to nonuniform systems with field variables, for example, to the time-dependent GL model, by generalizing the previous derivation[14] of the dynamic scaling law based on Kadanoff's cell analysis.[15]

APPENDIX A. MOMENTS IN THE INITIAL REGION

In the initial region, we have $y_{2n}(t) = O(\epsilon^n)$, as is seen from (22). Since $f_{2n}(\{y_{2j}(s)\})$ is a function of $y_{2n+2}, Y_{2n+4},\ldots,$ and $f_{2n}(0) = 0$, then $f_{2n}(\{y_{2j}(s)\}) = O(\epsilon^{n+1})$. Consequently, the first term in (25) does not affect $y_{2n}(t)$ to order ϵ^n. Furthermore, as is seen from the definition (20b), only the $c_2(x, \epsilon)$ in $g_{2n}(\{y_{2j}\})$ makes a contribution to $y_{2n}(t)$ to order ϵ^n. Thus, if we define $y_{2n}(t) = \epsilon^n a_n(t) + O(\epsilon^{n+1})$, then $a_n(t)$ is found to satisfy the following integral equation:

$$\epsilon^n a_n(t) = e^{2n\gamma t}\left[y_{2n}(0) + \epsilon^n n(2n-1)c_2(0,0)\int_0^t a_{n-1}(s)e^{-2n\gamma s}\,ds\right] \tag{A.1}$$

with $a_0(t) \equiv 1$. The solutions of these equations are obtained by mathematical induction as

$$a_n(t) = b_n\{\sigma(t)\}^n; \qquad \sigma(t) = \sigma e^{2\gamma t} - \sigma_1 \tag{A.2}$$

with (24). This proves (23).

APPENDIX B. DERIVATION OF (34)

From (20b), $g_{2n}(\{y_{2j}\}, \epsilon)$ can be expanded as

$$g_{2n}(\{y_{2j}\}, \epsilon) = n(2n-1)c_2(0,0)y_{2n-2} + R_{2n}(y_{2n}, y_{2n+2}, \ldots) \qquad \text{(B.1)}$$

It is easily shown that the remaining term R_{2n} does not make a finite contribution in (34), because $y_{2n}(\tau) = O(\tau^n)$ for a small τ. Therefore, we have

$$\begin{aligned}
&\lim_{\substack{\epsilon\to 0\\ \tau\ \text{fixed}}} \epsilon \int_{\tilde{\epsilon}}^{\tau} (2\gamma\xi^{n+1})^{-1} g_{2n}(\{y_{2j}\}, \epsilon)\, d\xi \\
&\quad = n(2n-1)c_2(0,0) \lim_{\substack{\epsilon\to 0\\ \tau\ \text{fixed}}} \epsilon \int_{\tilde{\epsilon}}^{\tau} (2\gamma\xi^{n+1})^{-1} y_{2n-2}(\xi, \epsilon)\, d\xi \\
&\quad = n(2n-1)c_2(0,0) \lim_{\substack{\epsilon\to 0, t\to 0\\ \tau\ \text{fixed}}} (\epsilon\tilde{\epsilon}^{-n}) \int_0^t e^{-2n\gamma s} y_{2n-2}(\sigma\epsilon e^{2\gamma s}, \epsilon)\, ds \\
&\quad = n(2n-1)c_2(0,0)\sigma^{-n} \int_0^\infty e^{-2n\gamma s} a_{n-1}(s)\, ds \\
&\quad = [1 - (\sigma_0/\sigma)^n] b_n
\end{aligned} \qquad \text{(B.2)}$$

where we have used the property that

$$y_{2n}(\sigma\epsilon e^{2\gamma t}, \epsilon) = \epsilon^n a_n(t) + O(\epsilon^{n+1}) \qquad \text{(B.3)}$$

with (A.2). Thus, we obtain (34) with the use of the relation

$$\tilde{\epsilon}^{-n} y_{2n}(0) = (\sigma_0/\sigma)^n b_n \qquad \text{(B.4)}$$

APPENDIX C. GENERATING FUNCTION AND DISTRIBUTION FUNCTION FOR A SYMMETRIC CASE

It is easily shown that the distribution function $\hat{P}(\xi, t)$ with $\xi = x^2$ satisfies the equation

$$\epsilon\, \partial\hat{P}(\xi, t)/\partial t + \hat{\mathscr{H}}(\xi, \epsilon\, \partial/\partial\xi, \epsilon)\hat{P}(\xi, t) = 0 \qquad \text{(C.1)}$$

for the symmetric case that

$$c_{2n}(-x, \epsilon) = c_{2n}(x, \epsilon) \quad \text{and} \quad c_{2n-1}(-x, \epsilon) = -c_{2n-1}(x, \epsilon) \qquad \text{(C.2)}$$

Here, $\hat{\mathscr{H}}$ is defined by

$$\hat{\mathscr{H}}(\xi, p, \epsilon) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \hat{c}_n(\xi, \epsilon) p^n \qquad \text{(C.3)}$$

Then, the generating function $\Psi(\lambda, t)$ corresponding to $\hat{P}(\xi, t)$ is proven to satisfy Eq. (38), by performing partial integrations iteratively.

APPENDIX D. DERIVATION OF THE SOLUTION $\Psi_{sc}(\lambda, t)$

It is easy to check that (42) is a solution of (36) that satisfies the boundary condition (41). In order to find the desired solution, we put

$$\Psi_{sc}(\lambda, \tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\exp -\frac{x^2}{2}\right) \exp[\lambda F(x, \tau)]\, dx \tag{D.1}$$

Then, $F(x, \tau)$ has to satisfy the following differential equation:

$$\gamma\tau\, \partial F/\partial\tau - \sqrt{F} c_1(\sqrt{F}) = 0 \tag{D.2}$$

The general solution of this equation is given by

$$F = [f^{-1}(\sqrt{\tau}\varphi(x))]^2 \tag{D.3}$$

where $\varphi(x)$ is an arbitrary function of x. The matching between (D.1) and (41) leads to the result that $\varphi(x) = x$. Thus, we obtain (42).

NOTE ADDED IN PROOF

The essence of this paper has already been shown to be valid even to multimacrovariables by the present author (to be published).

ACKNOWLEDGMENTS

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